Day 05

Rigid Body Transformations

Homogeneous Representation

- translation represented by a vector d
  - vector addition
- rotation represented by a matrix R
  - matrix-matrix and matrix-vector multiplication
- convenient to have a uniform representation of translation and rotation
  - obviously vector addition will not work for rotation
  - can we use matrix multiplication to represent translation?

Homogeneous Representation

- consider moving a point p by a translation vector d

$$p+d=\begin{bmatrix}p_x\\p_y\\p_z\end{bmatrix}+\begin{bmatrix}d_x\\d_y\\d_z\end{bmatrix}=\begin{bmatrix}p_x+d_x\\p_y+d_y\\p_z+d_z\end{bmatrix}$$

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}\begin{bmatrix}p_x\\p_y\\p_z\end{bmatrix}=\begin{bmatrix}p_x+d_x\\p_y+d_y\\p_z+d_z\end{bmatrix}$$

not possible as matrix-vector multiplication always leaves the origin unchanged

Homogeneous Representation

- consider an augmented vector p_h and an augmented matrix D

$$p_h = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Dp_h = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + d_x \\ p_y + d_y \\ p_z + d_z \\ 1 \end{bmatrix}$$

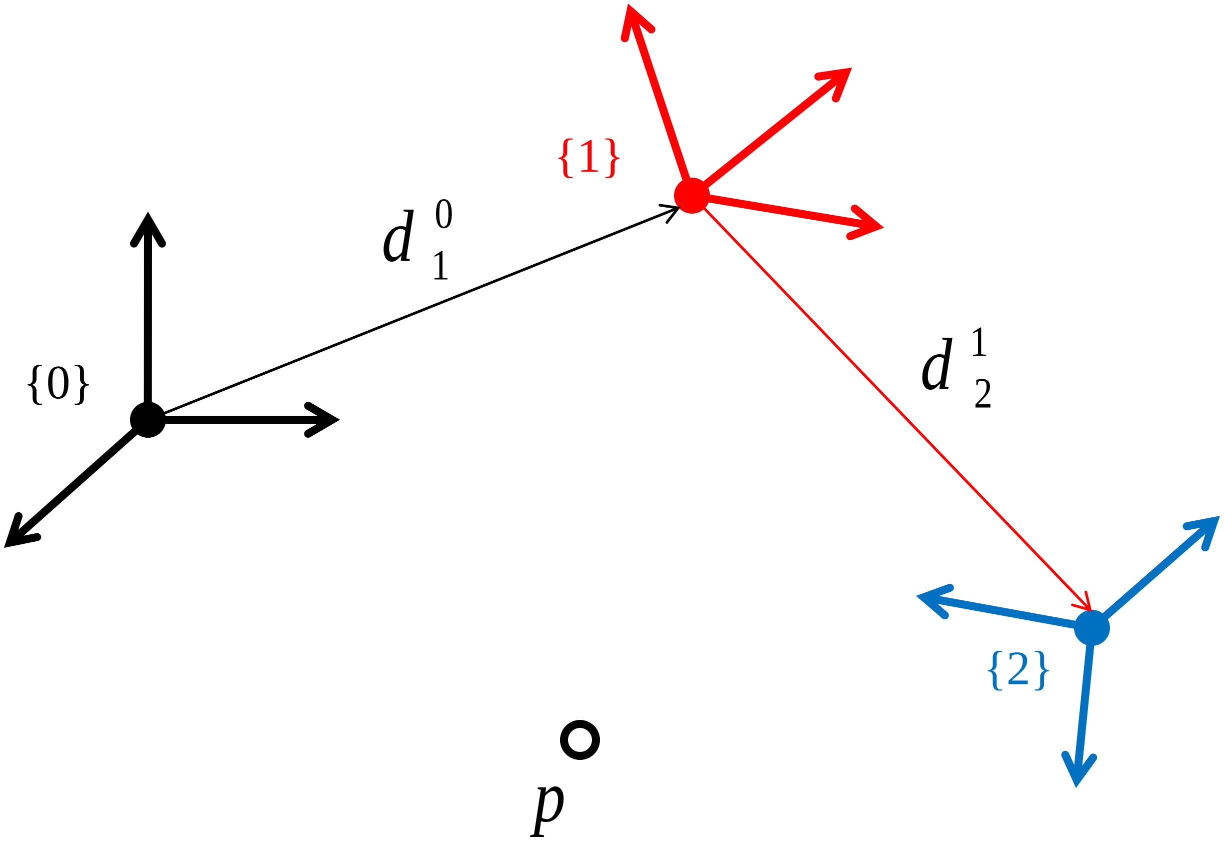
Homogeneous Representation

- the augmented form of a rotation matrix R_{3x3}

$$R = \begin{bmatrix} \begin{bmatrix} & & \\ & R_{3x3} & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

$$Rp_h = \begin{bmatrix} \begin{bmatrix} & & \\ & R_{3x3} & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ R_{3x3}p \\ \\ \end{bmatrix} \\ 1 \end{bmatrix}$$

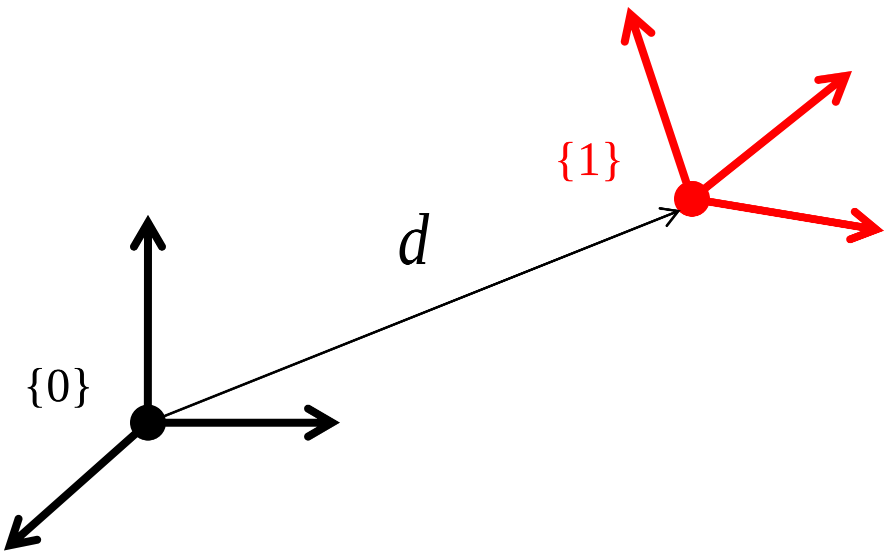
Rigid Body Transformations in 3D



Rigid Body Transformations in 3D

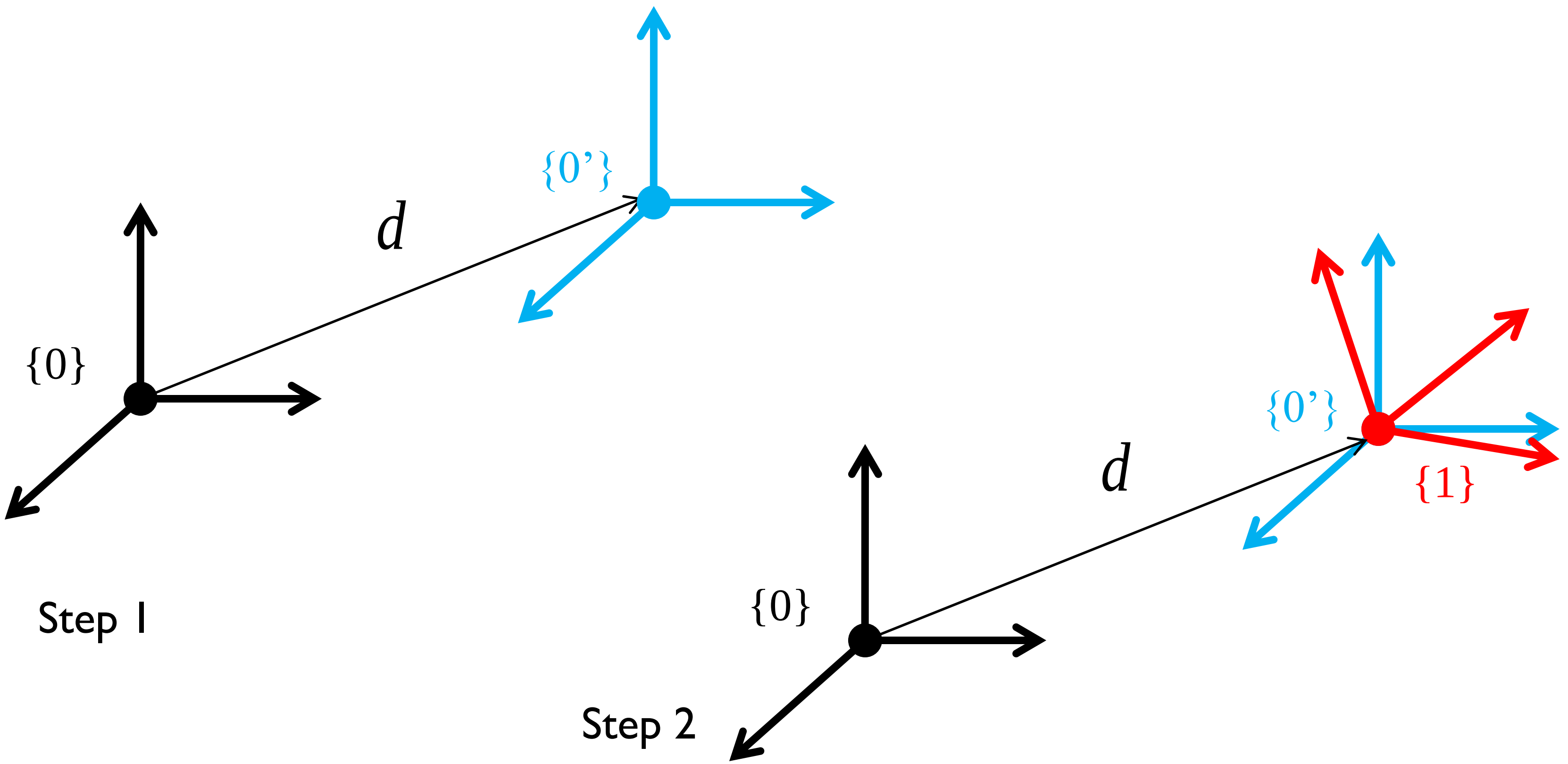
- suppose {1} is a rotated and translated relative to {0}
- what is the pose (the orientation and position) of {1} expressed in {0} ?

$$T_1^0 = ?$$



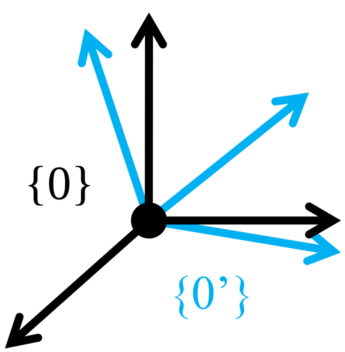
Rigid Body Transformations in 3D

- suppose we use the moving frame interpretation (postmultiply transformation matrices)
  1. translate in {0} to get {0'} $D_{0'}^{0}$
  2. and then rotate in {0'} to get {1} $D_{0'}^{0} R_{1}^{0'}$

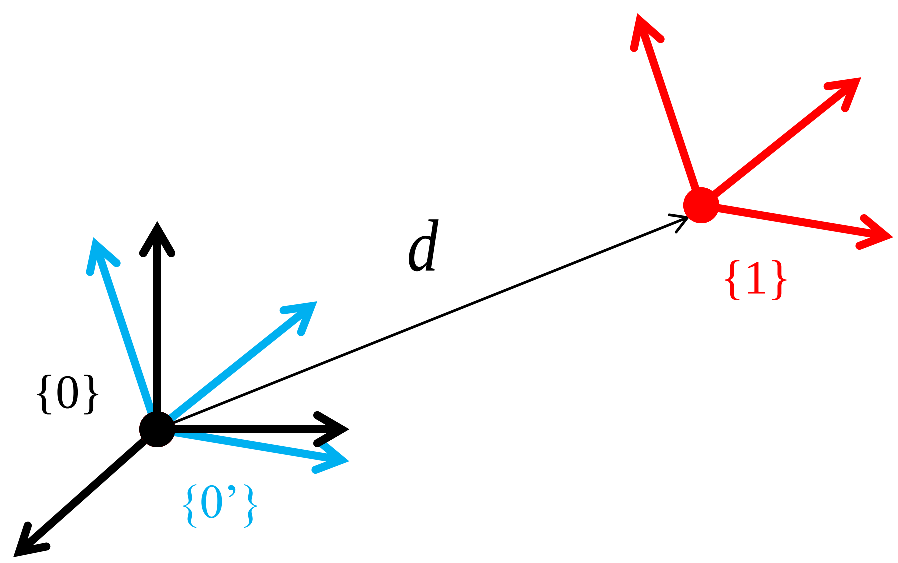


Rigid Body Transformations in 3D

- suppose we use the fixed frame interpretation (premultiply transformation matrices)
  1. rotate in {0} to get {0'} R
  2. and then translate in {0} in to get {1} $D\,R$



Step 1



Step 2

Rigid Body Transformations in 3D

- both interpretations yield the same transformation

$$T_1^0 = D\,R$$

$$= \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & R_{3x3} & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & R_{3x3} & & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Representation

- every rigid-body transformation can be represented as a rotation followed by a translation *in the same frame*
  - as a 4x4 matrix

$$T = \begin{bmatrix} & R & & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where R is a 3x3 rotation matrix and d is a 3x1 translation vector

Homogeneous Representation

- in some frame i
  - points

$$P^i = \begin{bmatrix} p^i \\ 1 \end{bmatrix}$$

  - vectors

$$V^i = \begin{bmatrix} v^i \\ 0 \end{bmatrix}$$

Inverse Transformation

- the inverse of a transformation undoes the original transformation
  - if

$$T = \begin{bmatrix} & R & & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - then

$$T^{-1} = \begin{bmatrix} & R^T & & -R^T d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$